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NATURAL FREQUENCIES OF AN INFINITE BEAM ON A SIMPLE INERTIAL FOUNDATION MODEL†

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Abstract—A simple inertial foundation model consisting of a 3-D shear layer resting on a bed of closely spaced individual rods capable of wave propagation in the axial direction is suggested here. The frequency equation for an infinite beam resting on such a foundation is determined. It is found to be the same as one given for a beam resting on a 3-D inertial layer. Therefore, this model can be successfully used for considering the effect of the inertia of the foundation on the dynamic response of the structures resting on it.

NOMENCLATURE

B	bending rigidity of the beam
b	width of the beam
c_0	speed of wave propagation in foundation rods
G	property of the shear layer
H	depth of foundation rods
k	elastic constant for Winkler springs
k_0	axial stiffness of foundation rods
m	mass per unit length of the beam
M_0	inertia of the vibrating load
p, p_1	foundation pressures
P	magnitude of the vibrating load
R	depth ratio
u	axial displacement of foundation rods
w	vertical displacement of the shear layer
w_1	vertical displacement of the beam
γ^*, h^*	non-dimensional frequency ratios
σ_z	normal stress in the foundation
ω_0	driving frequency.

1. INTRODUCTION

The natural frequency of a railroad track was first determined by Timoshenko[1]. He modeled the track as an infinite beam resting on a massless Winkler foundation. He arrived at a simple expression for the natural frequency $\omega_R = \sqrt{(k/m)}$, where k is the Winkler constant and m is the mass per unit length of the beam. Various other investigations about the vibrations of footings and strips have shown that the mass of the base has a significant effect on the dynamic response of the footings and strips[2–5]. Hence, it is essential to consider the mass of the supporting foundation for determining the natural frequencies of an infinite beam.

The natural frequencies of an infinite beam resting on a 3-D inertial elastic layer were determined in Ref. [6]. If the beam is very light as compared to the mass of the supporting foundation, the frequency equation was the same as that of a rod in axial vibration, which is fixed at the bottom and free at the top. It was observed that the resonance phenomenon occurred due to reflection of the waves from the boundary. This suggests consideration of a foundation model consisting of closely spaced individual rods, capable of wave propagation in the axial direction (like an inertial Winkler model). Therefore, the foundation is modeled as closely spaced individual rods (mass density ρ and elastic constant k) of length H, fixed at the bottom and connected at the top, to a 3-D shear layer with constant G

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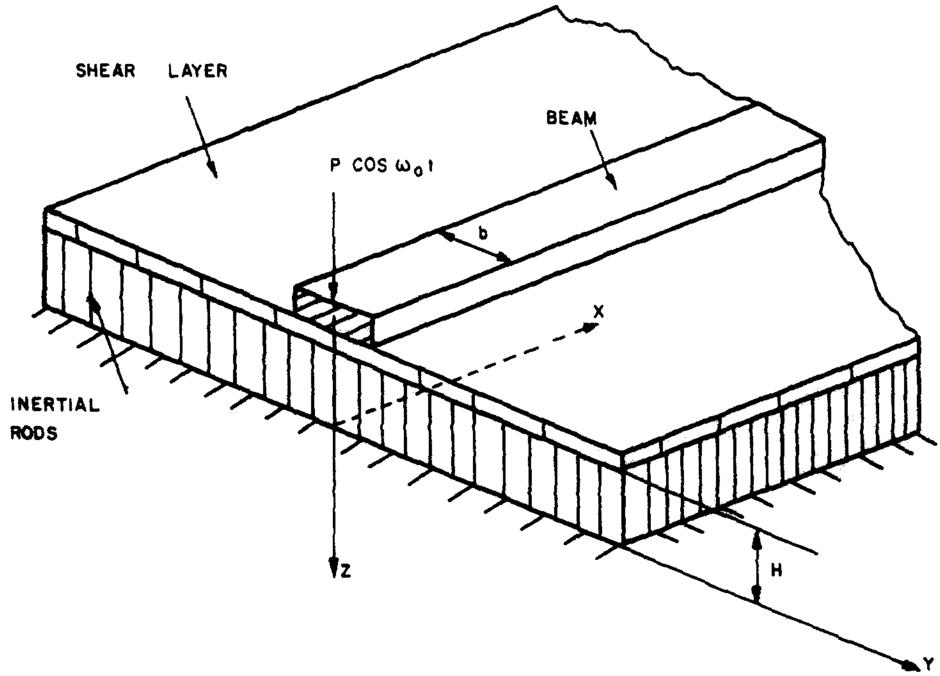

Fig. 1. Beam on a 3-D inertial foundation model.

(Fig. 1). The elastic constants for this model k and G, are determined by comparing the resulting frequency equation for the beam with the corresponding frequency equation given in Ref. [6]. As shown in Fig. 1, the beam is subject to a concentrated oscillating load $P \cos \omega_0 t$ with mass M_0.

2. DETERMINATION OF THE FOUNDATION RESPONSE

The pressure exerted by the foundation is

$$\bar{\sigma}_z(x,y,t) = -G\nabla^2\bar{w}(x,y,t)+\bar{p}_1(x,y,t) \tag{1}$$

where $\bar{p}_1(x,y,t)$ is the pressure exerted by the rods on the shear layer, and $\bar{w}(x,y,t)$ is the displacement of the shear layer. The motion of the "foundation rods" is governed by the familiar 1-D wave equation

$$\frac{\partial^2\bar{u}}{\partial t^2} = c_0^2\frac{\partial^2\bar{u}}{\partial z^2}$$
$$t>0 \quad \text{and} \quad 0 \leqslant z \leqslant H \tag{2}$$

where $\bar{u}(x,y,z,t)$ is the displacement of the rods, and c_0 is the velocity of wave propagation in the rod. The boundary and the matching conditions are

$$\bar{u}(x,y,0,t) = 0 \quad \text{and} \quad \bar{u}(x,y,H,t) = \bar{w}(x,y,t). \tag{3}$$

It is assumed that the following equations are valid for large time t[7]:

$$\begin{aligned}\bar{w}(x,y,t) &= w(x,y)\cos\omega_0 t\\ \bar{u}(x,y,z,t) &= u(x,y,z)\cos\omega_0 t\\ \bar{\sigma}_z(x,y,t) &= \sigma_z(x,y)\cos\omega_0 t\\ \bar{p}_1(x,y,t) &= p_1(x,y)\cos\omega_0 t.\end{aligned} \tag{4}$$

Substitution of the above equations into eqns (1)–(3) results in the elimination of the variable t. They are written as

$$\sigma_z = -G\nabla^2 w + p_1 \tag{5}$$

$$\frac{d^2u}{dz^2} + \frac{\omega_0^2}{c_0^2}u = 0 \tag{6}$$

$$u(x,y,0) = 0 \quad \text{and} \quad u(x,y,H) = w(x,y). \tag{7}$$

Equation (6) with the boundary and the matching conditions, eqn (7), is then solved. The result is

$$u(x,y,z) = \frac{\sin \omega_0 z/c_0}{\sin \omega_0 H/c_0} w(x,y). \tag{8}$$

The corresponding expression for the pressure $p_1(x,y)$, is given as

$$p_1(x,y) = k_0\left(\frac{du}{dz}\right)_{z=H} \tag{9}$$

where k_0 is the foundation constant (axial stiffness of the rods).

Substitution of eqn (8) into eqn (9) results in the pressure expression

$$p_1(x,y) = \frac{k_0\omega_0}{c_0} \cot\left(\frac{\omega_0 H}{c_0}\right) w(x,y). \tag{10}$$

If the rods are non-inertial, then eqn (10) reduces to the familiar expression for the Winkler model

$$p_1(x,y) = kw(x,y) \tag{11}$$

where k is the elastic constant for the Winkler springs. Noting the above result, eqn (10) is rearranged and written as

$$p_1(x,y) = k\gamma^* \cot(\gamma^*) w(x,y) \tag{12}$$

where $\gamma^* = \omega_0 H/c_0$ is the non-dimensional frequency ratio. Substitution of the above equation into eqn (5) results in the relationship between the pressure exerted by the foundation σ_z, and the displacement of the shear layer w

$$\sigma_z = -G\nabla^2 w + k\gamma^* \cot(\gamma^*) w. \tag{13}$$

Next, we assume that the solution for the displacement, w, can be expressed in the form of a symmetric double Fourier integral

$$w(x,y) = \int_0^\infty \int_0^\infty A_1' \cos \alpha x \cos \beta y \, d\alpha \, d\beta. \tag{14}$$

Substitution of the above equation into eqn (13) results in the expression for the pressure σ_z

$$\sigma_z(x,y) = \int_0^\infty \int_0^\infty [G(\alpha^2+\beta^2)+k\gamma^* \cot(\gamma^*)]A'_1 \cos\alpha x \cos\beta y \, d\alpha \, d\beta \tag{15}$$

where, the only unknown is A'_1.

3. MATCHING THE BEAM RESPONSE WITH THE FOUNDATION RESPONSE

The constant $A'_1(\alpha)$ is determined by matching the response of the beam with the response of the foundation. The equation of motion for the beam is

$$B\frac{\partial^4 \bar{w}_1}{\partial x^4} + m\frac{\partial^2 \bar{w}_1}{\partial t^2} = \left[P\cos\omega_0 t - M_0\frac{\partial^2 \bar{w}_1}{\partial t^2}\right]\delta(x) - \bar{p}(x,t) \tag{16}$$

where $\bar{w}_1(x,t)$ is the vertical displacement of the beam, B the bending rigidity, m the mass per unit length, and $\bar{p}(x,t)$ is the corresponding pressure exerted on the beam by the foundation. Assuming that the response quantities are in phase with the foundation response (eqn (4)); i.e.

$$\begin{aligned} \bar{w}_1(x,t) &= w_1(x)\cos\omega_0 t \\ \bar{p}(x,t) &= p(x)\cos\omega_0 t \end{aligned} \tag{17}$$

the steady state equation of motion becomes

$$B\frac{d^4 w_1}{dx^4} - m\omega_0^2 w_1 = [P+M_0\omega_0^2 w_1]\delta(x) - p(x). \tag{18}$$

The response of the beam, as well as the pressure distribution beneath the beam, is expressed in the form of a single Fourier integral

$$w_1(x) = \int_0^\infty C(\alpha)\cos\alpha x \, d\alpha \tag{19a}$$

$$p(x) = \frac{1}{\pi}\int_0^\infty A(\alpha)\cos\alpha x \, d\alpha. \tag{19b}$$

It is also assumed that the pressure distribution beneath the beam of width b, is uniform. Hence

$$p(x) = -b\sigma_z(x,0). \tag{20}$$

Noting eqn (19b), it follows that the pressure distribution beneath the beam is

$$\sigma_z(x,0) = \frac{1}{\pi}\int_0^\infty A_1(\alpha)\cos\alpha x \, d\alpha \tag{21}$$

where $A_1(\alpha) = -A(\alpha)/b$. Assuming an even rectangular pressure distribution under the beam in the y-direction, the above expression for the entire x–y plane may be written as a double Fourier integral (Ref. [6], p. 68)

$$\sigma_z(x,y) = -\frac{2}{\pi^2}\int_0^\infty \int_0^\infty \frac{A_1 \sin\frac{\beta b}{2} \cos \alpha x \cos \beta y}{\beta} \, d\alpha \, d\beta. \quad (22)$$

Matching the pressure expression given above with eqn (15), we have

$$A'_1 = -\frac{2 \sin \frac{\beta b}{2} A_1}{\pi^2 \beta[G(\alpha^2+\beta^2)+k\gamma^* \cot(\gamma^*)]}. \quad (23)$$

Substitution of eqn (23) into eqn (14) determines the displacement of the beam at its center line ($y = 0$)

$$w(x,0) = -\frac{2}{\pi^2}\int_0^\infty \int_0^\infty \frac{A_1 \cos \alpha x \sin \frac{\beta b}{2}}{\beta[G(\alpha^2+\beta^2)+k\gamma^* \cot(\gamma^*)]} \, d\alpha \, d\beta \quad (24)$$

which is rewritten as

$$w(x,0) = -\frac{2}{\pi^2 G}\int_0^\infty \frac{A_1 T(q) \cos \alpha x}{\alpha^2} \, d\alpha \quad (25)$$

where

$$T(q) = \int_0^\infty \frac{\sin q\tau}{\tau\left[1+\tau^2+\dfrac{kb^2\gamma^* \cot(\gamma^*)}{4Gq^2}\right]} \, d\tau \quad (26)$$

where $q = \alpha b/2$ and $\tau = \beta/\alpha$ are the new non-dimensional variables. The integral in eqn (26) is evaluated using tables of integrals (Ref. [8], p. 408)

$$T(q) = \frac{\pi}{2} \frac{1-\exp\left[-q\sqrt{\left(1+\dfrac{kb^2\gamma^* \cot(\gamma^*)}{4Gq^2}\right)}\right]}{1+\dfrac{kb^2\gamma^* \cot(\gamma^*)}{4Gq^2}}. \quad (27)$$

Substitution of eqn (19) into eqn (18) results in the relationship between A_1 and C

$$C(\alpha) = \frac{P+M_0\omega_0^2 w_1(0)+bA_1(\alpha)}{\pi B[\alpha^4 - m\omega_0^2/B]}. \quad (28)$$

Use of the above expression for $C(\alpha)$ in eqn (19a) results in the deflection of the beam as

$$w_1(x) = \int_0^\infty \frac{[P+M_0\omega_0^2 w_1(0)+bA_1(\alpha)] \cos \alpha x}{\pi B(\alpha^4 - m\omega_0^2/B)} \, d\alpha. \quad (29)$$

Next, the unknown A_1 is determined by matching the deflection of the beam and the displacement of the foundation (eqn (25)) at the surface of contact. The result is

$$A_1(\alpha) = -\frac{[P+M_0\omega_0^2 w_1(0)]\alpha^2 b/2}{2[\varepsilon T(q)(q^4-\delta)+q^2]} \tag{30}$$

where

$$\varepsilon = \frac{B(2/b)^3}{\pi G}$$

$$\delta = \frac{m\omega_0^2(b/2)^4}{B}. \tag{31}$$

Using the expression for $A_1(\alpha)$ in eqn (25), the displacement of the beam is found as

$$w_1(x) = \frac{P+M_0\omega_0^2 w_1(0)}{\pi^2 G}\int_0^\infty \frac{T(q)\cos 2qx/b}{q^2+\varepsilon T(q)[q^4-\delta]}\,dq. \tag{32}$$

Solving the above equation for the displacement under the load, we have

$$w_1(0) = \frac{PF^*(\omega_0)}{1-M_0\omega_0^2 F^*(\omega_0)} \tag{33}$$

where

$$F^*(\omega_0) = \frac{1}{\pi^2 G}\int_0^\infty \frac{T(q)}{q^2+\varepsilon T(q)[q^4-\delta]}\,dq. \tag{34}$$

The expression for the displacement under the load with the mass M_0, is given in the above equation, which consists of an integral. Next, the natural frequencies are determined by noting the frequencies at which the above equation becomes infinite.

4. THE NATURAL FREQUENCIES OF THE BEAM WHEN THE LOAD HAS MASS M_0

It is evident from eqn (33) that the displacement will become infinite if

$$M_0\omega_0^2 F^*(\omega_0) = 1 \tag{35}$$

which is the equation for determining the natural frequencies of the beam. If the mass of the vibrating load, M_0, is negligible, then the displacement will become infinite if $F^*(\omega_0) \to \infty$. It is fairly easy to show that the integral, $F^*(\omega_0)$, becomes infinite near the left end point, $q = 0$, if (Ref. [6], p. 71)

$$\lim_{q\to 0} \varepsilon\delta T(q) = q^2. \tag{36}$$

Using eqn (27), the behaviour of $T(q)$ for small values of q is determined as

$$\lim_{q\to 0} T(q) = \frac{2\pi G q^2}{kb^2\gamma^*\cot(\gamma^*)}\left\{1-\exp\left[-\frac{b}{2}\sqrt{\left(\frac{k\gamma^*\cot(\gamma^*)}{G}\right)}\right]\right\}. \tag{37}$$

Substitution of eqns (37) and (31) into eqn (36) results in the following frequency equation:

$$m\omega_0^2 = \frac{kb\gamma^* \cot(\gamma^*)}{1-\exp\left[-\frac{b}{2}\sqrt{\left(\frac{k\gamma^* \cot(\gamma^*)}{G}\right)}\right]}. \tag{38}$$

Using the standard definition of the velocity of wave propagation in the rods

$$c_0 = \sqrt{\left(\frac{k_0}{\rho}\right)} \tag{39}$$

where k_0 is the foundation stiffness and ρ is the mass density of the foundation, eqn (38) may be rewritten in the non-dimensional form as

$$M^*\gamma^* = \frac{\cot(\gamma^*)}{1-\exp\left[-\frac{b}{2}\sqrt{\left(\frac{k\gamma^* \cot(\gamma^*)}{G}\right)}\right]} \tag{40}$$

where

$$M^* = m/\rho bH.$$

If the parameters k, G, and k_0 are chosen as

$$\begin{aligned} k &= (\lambda+2\mu)/H \\ k_0 &= \lambda+2\mu \\ G &= kH^2/2 \end{aligned} \tag{41}$$

then eqn (40) becomes

$$M^*\gamma^* = \frac{\cot(\gamma^*)}{1-\exp\left[-\frac{b}{2}\sqrt{\left(\frac{2\gamma^* \cot(\gamma^*)}{H^2}\right)}\right]}. \tag{42}$$

After some rearrangement, the above equation can also be written in the form

$$M^*\gamma^* = \frac{\cot(\gamma^*)}{1-\exp\left[-h^*\sqrt{\left(\frac{2\cot(\gamma^*)}{\gamma^*}\right)}\right]} \tag{43}$$

which is exactly the same as the frequency equation obtained for an infinite beam resting on a 3-D inertial layer (Ref. [6], p. 73). It should be noted that the frequency variable γ^* is the same as the variable f used in Ref. [6] and $h^* = \gamma^* b/2H$. Thus, the model (Fig. 1) used here is satisfactory as far as the determination of the natural frequencies is concerned.

Next, we determine the frequency equation for the beam when the load has mass M_0, by using the parameters given in eqn (41). Equation (35) is written in the non-dimensional form as

$$RM'h^{*2}F^*(h^*) = \frac{\pi^2}{8} \tag{44}$$

where $h^* = b\gamma^*/2H$, $M' = M_0/\rho b^3$, and $F^*(h^*)$ is defined as

$$F^*(h^*) = \int_0^\infty \frac{T(q)}{q^2[1+\varepsilon q^2 T(q)]}\,dq \tag{45}$$

where the mass of the beam m, is neglected and

$$T(q) = \frac{\pi}{2}\,\frac{1-\exp\left[-q\sqrt{\left(1+\frac{Rh^*\cot(2h^*/R)}{q^2}\right)}\right]}{1+\frac{Rh^*\cot(2h^*/R)}{q^2}}. \tag{46}$$

5. NUMERICAL RESULTS AND COMPARISON WITH EXPERIMENTAL DATA

In this section, validity of the frequency equation, eqn (44), is examined by comparing the natural frequencies obtained from it with those found by Schneider[9] and Ono and Itoo[10] from their experiments.

Schneider[9] has given a detailed account of the investigations carried out at the track test bench of the Institute of Ground Transportation of the Technical University of Munich. The railroad track consisted of the rails and nine ties laid on the layers of ballast and sand in a box. The track was then excited by a vibrator with an equivalent unsprung weight of 3000 kg. The resulting dynamic stiffness was noted. The resonant frequencies were the ones at which the dynamic stiffness was minimum. For the given experimental setup two resonant frequencies were recorded as $f_1 = 30$ Hz and $f_2 = 78$ Hz. Since the range of oscillation was limited to 0–100 Hz, higher natural frequencies were not recorded.

The other parameters for Schneider's experiment are[9]; the mass per unit length of the rails and ties $m = 76$ kg s^2 m^{-2}, the mass density of the foundation $\rho = 196$ kg s^2 m^{-4}, the mass of the vibrating load $M_0 = 3000/9.81 = 312$ kg s^2 m^{-1}, the modulus of elasticity of the foundation $E = 0.85\times 10^7$ kg m^{-2}, Poisson's ratio $\nu = 0.3$, the width of the ties $b = 2.6$ m, and the depth of the layer $H = 2.3$ m.

Using these values, other parameters are found, as, the depth ratio $R = b/H = 1.13$, the mass ratio $M' = M_0/\rho b^3 = 0.09$, Lamê constants $\lambda = 0.49\times 10^7$ kg m^{-2}, and $\mu = 0.327\times 10^7$ kg m^{-2}.

Inserting these parameters into eqns (45) and (46) and solving eqn (44) numerically for the first two roots yields the non-dimensional frequencies, $h_1^* = 0.887$ and $h_2^* = 2.531$.

The corresponding natural frequencies are

$$f_1 = \frac{h_1^*}{\pi b}\sqrt{((\lambda+2\mu)/\rho)} = 26.24 \text{ Hz}$$

$$f_2 = \frac{h_2^*}{\pi b}\sqrt{((\lambda+2\mu)/\rho)} = 74.86 \text{ Hz}.$$

Therefore, the natural frequencies obtained using the theory developed here agree closely with the experimental results of Schneider.

Ono and Itoo[10] studied the vibrations of a test track of 8 m length which was laid using 50 kg rails, prestressed concrete ties and crushed stone ballast. The track was then excited using a vibrator with an equivalent unsprung weight of 1800 kg. The range of oscillations of the vibrator was 800–2500 r.p.m. The recorded resonant frequencies were $f_1 = 19$ Hz and $f_2 = 29.58$ Hz.

The other parameters are[10]; the mass per unit length of the track $m = 76$ kg s^2 m^{-2}, the mass density of the foundation $\rho = 196$ kg s^2 m^{-4}, the mass of the vibrating load $M_0 = 1800/9.81 = 183$ kg s^2 m^{-1}, Poisson's ratio $\nu = 0.3$, and the width of the ties $b = 2.0$ m. Hence, the mass ratio $M' = M_0/\rho b^3 = 0.117$.

The modulus of elasticity for the foundation material and the depth of the foundation were not provided by Ono and Itoo[10]. Therefore, the values for these parameters are assumed so that one of the theoretical frequencies coincides with the corresponding experimental value. The assumed parameters are: the modulus of elasticity of the layer $E = 0.65 \times 10^7$ kg m^{-2}, and the depth of the subgrade $H = 8.5$ m. Hence, the depth ratio $R = b/H = 0.235$, Lamê constants $\lambda = 0.375 \times 10^7$ kg m^{-2}, and $\mu = 0.25 \times 10^7$ kg m^{-2}.

Inserting these parameters into eqns (45) and (46) and solving eqn (44) numerically for the first three roots results in $h_1^* = 0.185$, $h_2^* = 0.554$, and $h_3^* = 0.923$.

The corresponding natural frequencies are

$$f_1 = \frac{h_1^*}{\pi b}\sqrt{((\lambda+2\mu)/\rho)} = 6.22 \text{ Hz}$$

$$f_2 = \frac{h_2^*}{\pi b}\sqrt{((\lambda+2\mu)/\rho)} = 18.63 \text{ Hz}$$

$$f_3 = \frac{h_3^*}{\pi b}\sqrt{((\lambda+2\mu)/\rho)} = 31.04 \text{ Hz.}$$

It is of interest to note that the second and the third natural frequency agrees very well with the corresponding experimental value of Ono and Itoo. The first natural frequency was not recorded by them because it fell outside the frequency range of the excitor.

6. CONCLUSIONS

A simple inertial model is presented for the dynamic analysis of an infinite beam resting on an inertial layer. The resulting frequency equation is the same as the one for an actual layer.

The frequency equation for determining the natural frequencies of the beam with the concentrated mass at its center, is provided. The natural frequencies, determined by solving this equation, agreed very closely with experimental results[9, 10].

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